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The non-linear oscillations of a rod with close values of the cross-section axial moments of inertia☆

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ABSTRACT

An analytical solution of the problem of the forced flexural oscillations of a rod with fixed hinged supports is presented. The rod has close natural frequencies of flexural oscillations in two mutually perpendicular planes due to the close values of the principal axial moments of inertia of the cross-section. The geometrical non-linearity, due to the change in the length of the middle line of the rod when it undergoes three-dimensional motion, is taken into account. The oscillations of the rod in the neighbourhood of the principal and first superharmonic resonances are investigated.

In the classical problem of the non-linear oscillations of a string and a rod with fixed hinged supports,[1] the motions of the system in one plane are considered. Such systems have a strict geometrical non-linearity. In investigations of the three-dimensional oscillations of similar systems[2–5] it has been found that the oscillations in different directions are interrelated, leading to the existence of both two-dimensional and three-dimensional forms of motion. For forced oscillations in the neighbourhood of the principal resonance there is a range of frequencies at which stable parametric oscillations occur in a plane orthogonal to the action of the inducing force, and the resultant motion of points occurs along an ellipse.

For the oscillations of an inextensible and elastic thread with a stretching device, there are also two-dimensional and three-dimensional forms of motion.[6,7]

The free oscillations of a rod with fixed hinged supports and close values of the axial moments of inertia were investigated in Ref.5 and the results of a numerical solution of the problem of forced oscillations in the neighbourhood of the principal resonance were given taking dissipation into account. Below we give an analytical solution of the problem of forced oscillations of a rod which can be obtained ignoring dissipation. The corresponding amplitude-frequency characteristics, obtained numerically and taking dissipation into account, are presented.

1. Formulation of the problem

Suppose the central axis of the rod in the undeformed state coincides with the x axis of a rectangular system of coordinates, and the principal axes of inertia of the cross section are parallel to the y and z axes. The coordinates $x = 0$ and $x = L$ correspond to the ends of the rod. We will denote the displacements of points of the middle line of the rod by v and w.

The equations of three-dimensional non-linear oscillations of the rod, ignoring dissipation, obtained previously,[5] have the form

$$\pi^4\ddot{v} + v^{IV} - 4\gamma_1\varepsilon_0 v'' = 0, \quad \pi^4\ddot{w} + cw^{IV} - 4\gamma_1\varepsilon_0 w'' = 0$$
$$\varepsilon_0 = \frac{1}{2}\int_0^1 (v'^2 + w'^2)dx, \quad \gamma_1 = \frac{FL^2}{4J_y}, \quad c = \frac{J_z}{J_y} \tag{1.1}$$

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All the displacements and the x coordinate are normalized to the length of the rod L. The dots indicate differentiation with respect to the dimensionless time

$$t_* = \omega_0 t; \quad \omega_0 = (\pi/L)^2 (EJ_y)^{1/2} (\rho F)^{-1/2}$$

Here ω_0 is the first natural frequency of flexural oscillations in the Oxz plane, E and ρ are the modulus of elasticity and the density of the rod material, F is the cross-section area, J_y and J_z are the moments of inertia of the cross section, and ε_0 is the strain of the middle line. The last equation of (1.1) is obtained from the condition for there to be no relative longitudinal displacement of the supports. Torsion is ignored, since we will henceforth mainly consider flexural oscillations of a thin rod in the neighbourhood of the first natural frequency.

For a rod with fixed hinged supports at the ends, the boundary conditions have the form

$$\upsilon = w = \upsilon'' = w'' = 0 \quad \text{for} \quad x = 0, L \tag{1.2}$$

Suppose external harmonic loads act on the rod in mutually perpendicular directions $q_\upsilon^*(x, t)$ and $q_w^*(x, t)$, and dissipation is taken into account using models of internal and external friction. The equations of forced oscillations of the rod in dimensionless variables have the form

$$\begin{aligned} &\pi^4 \ddot{\upsilon} + \upsilon^{IV} + \beta \dot{\upsilon}^{IV} + \pi^4 \sigma \dot{\upsilon} - 4\gamma_1 \varepsilon_0 \upsilon'' = q_\upsilon(x, t) \\ &\pi^4 \ddot{w} + c w^{IV} + c\beta \dot{w}^{IV} + \pi^4 \sigma \dot{w} - 4\gamma_1 \varepsilon_0 w'' = q_w(x, t) \end{aligned} \tag{1.3}$$

Here

$$\beta = \beta_* \omega_0, \quad \sigma = \frac{\sigma_*}{\omega_0}, \quad q_\upsilon = \frac{q_\upsilon^* L^3}{EJ_y}, \quad q_w = \frac{q_w^* L^3}{EJ_y}$$

and β_* and σ_* are the coefficients in the model of internal and external friction.

We will confine ourselves to the case of the single-mode approximation and we will represent the solution in a form which satisfies boundary conditions (1.2)

$$\upsilon(x, t) = \varphi_1(t) \sin \pi x, \quad w(x, t) = \varphi_2(t) \sin \pi x$$

Substituting into Eq. (1.3) we obtain a system with two degrees of freedom

$$\ddot{\varphi}_k + \varepsilon \eta \dot{\varphi}_k + [1 + (k-1)\varepsilon\delta] \varphi_k + \varepsilon\gamma(\varphi_1^2 + \varphi_2^2)\varphi_k = \varepsilon f_k \cos(\mu t + \psi_k), \quad k = 1, 2 \tag{1.4}$$

where we have introduced the small parameter ε and the notation

$$c = 1 + \varepsilon\delta, \quad \varepsilon\eta = \beta + \sigma, \quad \varepsilon\gamma = \gamma_1, \quad \varepsilon f_k \cos(\mu t + \psi_k) = 2\int_0^1 q_k(x, t) \sin \pi x dx$$

The non-linearity of the system, the amplitudes of the loads, the friction forces and the difference of the natural frequencies are assumed to be asymptotically small, which enables us to use the effective methods of non-linear mechanics.[8,9]

2. Oscillations in the neighbourhood of the principal resonance

We will represent system (1.4) in standard form by changing to slow variables

$$\varphi_k = b_k \cos(t + \alpha_k), \quad k = 1, 2 \tag{2.1}$$

where b_k and α_k are the amplitudes and phases of the partial oscillations, and the oscillations are considered in a small neighbourhood of the single frequency $\mu = 1 + \varepsilon\lambda$. Using the averaging method, we obtain the following system of equations in the slow variables

$$\begin{aligned} \dot{b}_k &= \frac{1}{2}\varepsilon f_k \sin(\alpha_k - \psi_k) + \frac{1}{8}\varepsilon\gamma b_1^k b_2^{3-k} \sin 2(\alpha_k - \alpha_{3-k}) - \frac{1}{2}\varepsilon\eta b_k \\ \dot{\alpha}_k &= \frac{1}{2}\varepsilon[\delta(k-1) - 2\lambda] - \frac{1}{2b_k}\varepsilon f_k \cos(\alpha_k + \psi_k) + \\ &+ \frac{3}{8}\varepsilon\gamma b_k^2 + \frac{1}{8}\varepsilon\gamma b_{3-k}^2 (2 + \cos 2(\alpha_1 - \alpha_2)); \quad k = 1, 2 \end{aligned} \tag{2.2}$$

We will further consider steady oscillations, which corresponds to zero left-hand sides of the equations. A numerical solution of the system of non-linear algebraic equations (2.2) can be obtained by analytical continuation of the solution with respect to a parameter.[10] Suppose that, for the system of equations $U_i(r,\lambda) = 0$ for a certain value of λ^k, we know the approximate solution $r^k = (b_1^k, b_2^k, \alpha_1^k, \alpha_2^k)$. Then, for the value $\lambda^{k+1} = \lambda^k + \Delta\lambda$ the approximate solution can be represented in the form $r^{k+1} = r^k + \Delta r$. Substituting into system (2.2) and linearizing the equations obtained, we obtain increments of the unknowns from the system

$$G\Delta r = p^k + R^k; \quad p^k = (0, 0, \lambda^k, \lambda^k)^T \tag{2.3}$$

where R^k is the discrepancy vector at the previous step of the solution, and the elements of the matrix G have the form $g_{ij} = dU_i/dr$.

To investigate the stability of the solutions using Lyapunov's second method we will consider a certain perturbed solution of system of equations (2.2) $\hat{r}(t) = r(t) + \Delta r(t)$. After substitution into the system and linearization, we obtain the equations of perturbed motion of the first approximation $\Delta\dot{r} = G\Delta r$. The sign of the real part of all the eigenvalues of the matrix G enables us to determine whether the solution is stable.

The solution of system of non-linear equations (2.2) is reduced to solving a sequence of systems of linear equations (2.3). At each step of the calculations we monitor the value of the discrepancy, and if the relative error exceeds a specified accuracy, the step of the varied variable is reduced. In the numerical solution we investigate the convergence for different values of the specified calculation accuracy. At points where the solutions branch we take as the independent parameter the variable with the increment of greatest modulus in the previous step, which enables all the important solutions to be obtained and enables multivalued amplitude-frequency and phase-frequency characteristics to be constructed.

We can obtain an analytical solution of system (2.2) when there are no dissipative forces. We will consider the important practical case of the oscillations of a rod due to the action of a load lying in one plane, for example, in the z plane. We will introduce the additional assumption that a small component of the load $f_1 \ll f_2$ exists in the y direction. In practice such a component is always present since no source of harmonic load is ideal. The presence on the right-hand sides of Eqs (2.2) of terms with the factor f_k/b_k does not enable us to obtain solutions corresponding to the motions of the rod in a plane. We will therefore first obtain solutions for finite values of f_1 and then take the limit as $f_1 \to 0$.

Without loss of generality we can put $\psi_2 = 0$. One solution of the problem exists for $\psi_1 = \pi/2$ and values of the phase increments $\alpha_1 = \pi/2$ and $\alpha_2 = 0$; in this case the load vector describes an ellipse in the yz plane. The second solution corresponds to $\psi_1 = 0$, $\alpha_1 = 0$ and $\alpha_2 = 0$, in which case the load vector is in a certain plane at an angle to the y axis. In both cases it follows from the first two equations that $b_k = \text{const}$, $k = 1, 2$.

Consider the first solution. In this case the last two equations of (2.2) take the form

$$-\lambda + \frac{1}{2}(k-1)\delta + \frac{3}{8}\gamma b_k^2 + \frac{1}{8}\gamma b_{3-k}^2 - \frac{f_k}{2b_k} = 0, \quad k = 1, 2 \tag{2.4}$$

Eliminating the frequency difference λ from the system, we obtain a relation between the amplitudes and the next solutions:

$$b_2^2 - \frac{2f_2}{\gamma b_2} = b_1^2 - \frac{2\delta}{\gamma} - \frac{2f_1}{\gamma b_1}$$
$$\lambda = \frac{3}{8}\gamma b_1^2 + \frac{1}{8}\gamma b_2^2 - \frac{f_1}{2b_1} = \frac{\delta}{2} + \frac{1}{8}\gamma b_1^2 + \frac{3}{8}\gamma b_2^2 - \frac{f_2}{2b_2} \tag{2.5}$$

By varying the variable b_2, we can determine the second amplitude as the root of the cubic equation shown, after which the second equation of (2.5) defines the amplitude-frequency characteristics of the system. For each value of b_2 there is a maximum of three roots b_1 and three values of the frequency difference λ.

Solution (2.5) has one more simple form in special cases

$$b_1 = -\frac{2f_1 b_2}{\gamma b_2 \beta(b_2)}, \quad \lambda = \frac{\delta}{2} + \frac{3}{8}\gamma b_2^2 - \frac{f_2}{2b_2} \text{ when } b_1 \ll b_2$$
$$\beta(b_2) = b_2^2 + \frac{2\delta}{\gamma} - \frac{2f_2}{\gamma b_2} \tag{2.6}$$

which corresponds to the motion of a rod in practically the plane of action of the load, and

$$b_1 = \sqrt{\beta(b_2)}, \quad \lambda = \frac{3\delta}{4} + \frac{1}{2}\gamma b_2^2 - \frac{3f_2}{4b_2} \text{ when } b_1 \approx b_2 \tag{2.7}$$

which corresponds to the three-dimensional motion of the rod. Solutions (2.6) and (2.7) are identical with (2.5) everywhere, apart from a region adjacent to the point at which the condition $\beta(b_2) = 0$ is satisfied.

The matrix G consists of four 2×2 blocks, where the blocks G_{11} and G_{22} are zero blocks, while the remaining two have the form

$$G_{12} = \frac{1}{4}\begin{Vmatrix} -\gamma b_1 b_2^2 - 2f_1 & \gamma b_1 b_2^2 \\ \gamma b_1^2 b_2 & -\gamma b_1^2 b_2 - 2f_2 \end{Vmatrix}$$
$$G_{21} = \frac{1}{4}\begin{Vmatrix} 3\gamma b_1 + 2f_1/b_1^2 & \gamma b_2 \\ \gamma b_1 & 3\gamma b_2 + 2f_2/b_2^2 \end{Vmatrix} \tag{2.8}$$

Since the diagonal blocks of the matrix G are zero blocks, its eigenvalues are defined as the roots of the biquadratic equation

$$\lambda^4 + A\lambda^2 + \det G_{12} \det G_{21} = 0$$

where

$$A = g_{23}g_{32} + g_{24}g_{42} + g_{13}g_{31} + g_{14}g_{41}$$

The conditions for the solution to be stable have the form

$$A > 0, \quad 0 < 4\det G_{12}\det G_{21} < A^2$$

We will consider the second solution of system (2.2) for the special case when $\psi_1 = 0$, for values of the phase additions $\alpha_1 = 0$ and $\alpha_2 = 0$. In this case the last two equations take the form

$$-\lambda + \frac{1}{2}(k-1)\delta + \frac{3}{8}\gamma b_k^2 + \frac{3}{8}\gamma b_{3-k}^2 - \frac{f_k}{2b_k} = 0, \quad k = 1, 2 \tag{2.9}$$

Equations (2.9) have the unique solution

$$b_1 = \frac{f_1}{\xi(b_2)}, \quad \lambda = \frac{\delta}{2} + \frac{3}{8}\gamma b_2^2\left(1 + \frac{f_1}{\xi^2(b_2)}\right) - \frac{f_2}{2b_2}, \quad \xi(b_2) = -\delta + \frac{f_2}{b_2} \tag{2.10}$$

which is identical with (2.6) when $f_1 \to 0$. The matrix G in this case also has a partitioned form with zero blocks G_{11} and G_{22} and non-zero blocks

$$G_{12} = \frac{1}{4}\left\| \begin{matrix} \gamma b_1 b_2^2 - 2f_1 & -\gamma b_1 b_2^2 \\ -\gamma b_1^2 b_2 & \gamma b_1^2 b_2 - 2f_2 \end{matrix} \right\|$$

$$G_{21} = \frac{1}{4}\left\| \begin{matrix} 3\gamma b_1 + 2f_1/b_1^2 & 3\gamma b_2 \\ 3\gamma b_1 & 3\gamma b_2 + 2f_2/b_2^2 \end{matrix} \right\| \tag{2.11}$$

Hence, to construct the amplitude-frequency characteristics it is necessary to construct the solutions (2.5) and (2.10), to investigate their stability with respect to the eigenvalues of the matrices G (2.8) and (2.11) and take the limit as $f_1 \to 0$.

Without loss of generality, the parameters γ and f_2 can be put equal to unity by elementary transformations, and hence the results of the calculations are given below for these values of the parameters.

In Fig. 1 we show the solutions (2.2) and (2.10) in the form of curves of $b_2(\lambda)$ and $b_1(\lambda)$ for the following values of the parameters

$$\gamma = 1, \quad f_1 = 10^{-5}, \quad f_2 = 1, \quad \delta = 1/2$$

which corresponds to the excitation of oscillations of the rod in the plane of high flexural stiffness. The stable solutions are shown by the heavy curves. The points B_k on the $b_1(\lambda)$ graphs in the orthogonal plane correspond to the points A_k on the amplitude-frequency characteristics $b_2(\lambda)$ in the plane of excitation of the oscillations.

The curves with the numbers *1 - 6* correspond to solution (2.5). Solutions *1*, *2*, *3* and *6* are the motions in a plane, and on the graph of $b_1(\lambda)$ the lines on the abscissa axis correspond to them.

Solutions *1*, *2* and *3* correspond to the motion of points on the middle line of the rod, in practice in the plane of action of the load $f_2(b_1/b_2 \approx f_1/f_2)$, which is identical with solution (2.6) everywhere apart from a region adjoining the point A_1 (the point B_1 respectively). The point A_2 (B_2) on this curve separates the unstable and stable parts *2* and *3*. Solution *4* represents three-dimensional motions, for which the cross-section of the rod moves in an ellipse in the same direction as the motion of the load vector. Solution *5* is three-dimensional motion in the opposite direction. Solutions *4* and *5* are identical with solution (2.7) everywhere, apart from a region adjacent to the point A_1 (B_1). Curve *6* corresponds to motion beyond resonance in the plane of action of the load f_2, and solutions *7* and *8* are unstable. Solution (2.10) has a singularity at $b_2 = f_2/\delta$; two curves: *1*, *2*, *9* and *3*, *10* and the third curve *6* correspond to it. It is identical with solution (2.5) everywhere, apart from the point A_2 (B_2) and has unstable sections *9* and *10*, corresponding to the motion of the cross-section of the rod in a certain plane with a constant amplitude in the plane in which the oscillations are excited.

We will determine the point of stability change on curves *2* and *3*, for which we substitute approximate solution (2.6) into matrix (2.8). Since $\det G = \det G_{12}\det G_{21}$, the change in sign of each of the determinants of matrices (2.8) leads to a change in the stability of the solution.

The determinant of the matrix G_{12} has the form

$$\det G_{12} = -\frac{f_1 f_2 b_2^2}{4\beta(b_2)} + \frac{f_1^3 b_2}{2\gamma\beta^2(b_2)} + \frac{f_1 f_2}{4}$$

and when $f_1 \to 0$ the determinant vanishes if $b_1 = f_2/\delta$. From solution (2.6) we determine the corresponding frequency difference, which is also the coordinates of the point A_2

$$b_2 = \frac{f_2}{\delta}, \quad \lambda = \frac{3}{8}\gamma\frac{f_2^2}{\delta^2} \tag{2.12}$$

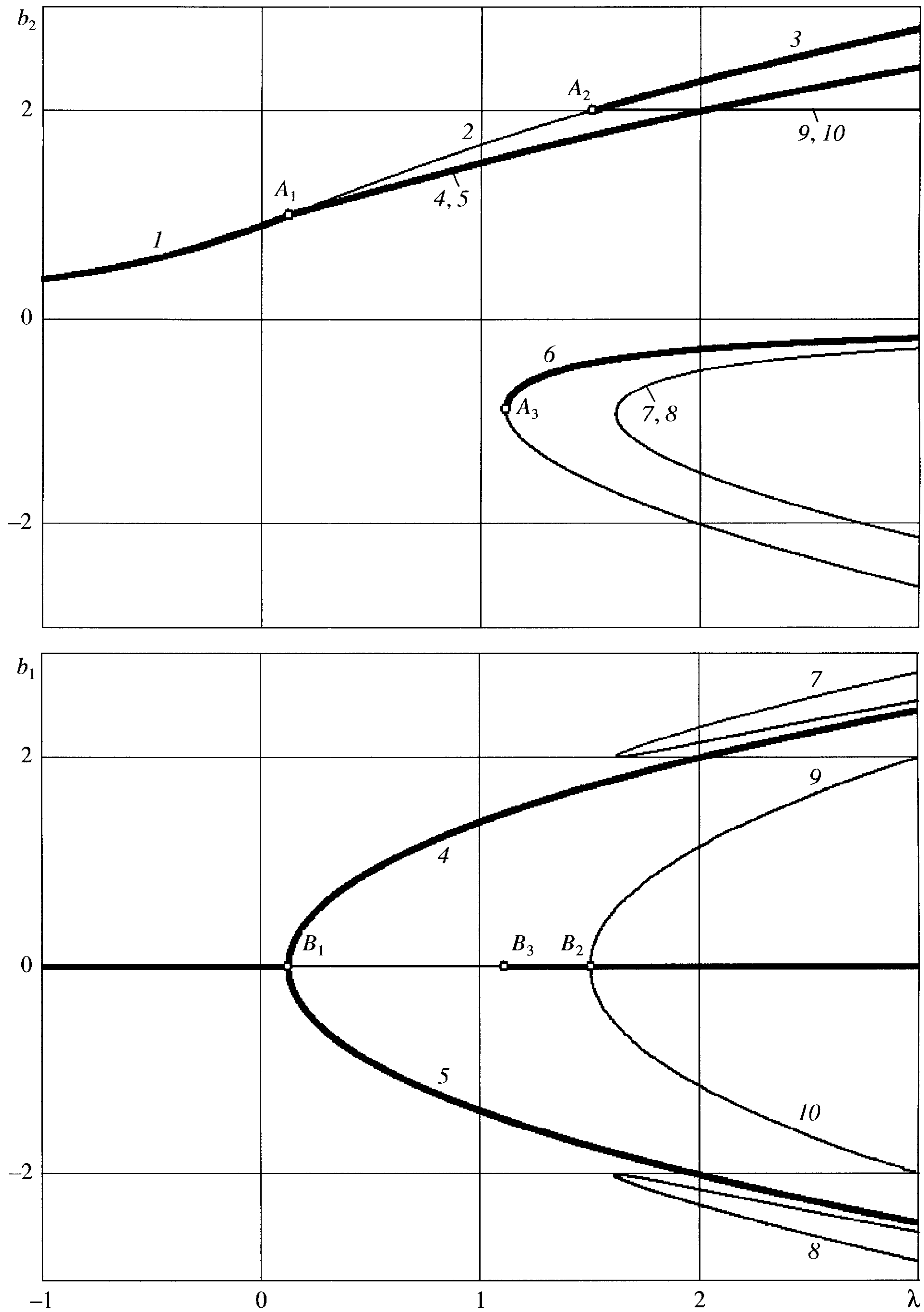

Fig. 1.

When the determinant

$$\det G_{21} = -\frac{f_1}{\beta(b_2)}\left(\gamma b_2 + \frac{f_2}{b_2^2}\right) + \frac{\gamma^2}{16 f_1}\left(\frac{3}{2}\gamma b_2 + \frac{f_2}{b_2^2}\right)\beta(b_2)$$

is equated to zero, when $f_1 \to 0$ we obtain the ordinate $b_2 = -(2f_2/(3\gamma))^{1/3}$ of the bifurcation point A_3 in the solution, to which curve *6* corresponds. The determinant also changes sign when $\beta(b_2) = 0$, which corresponds to bifurcation point A_1.

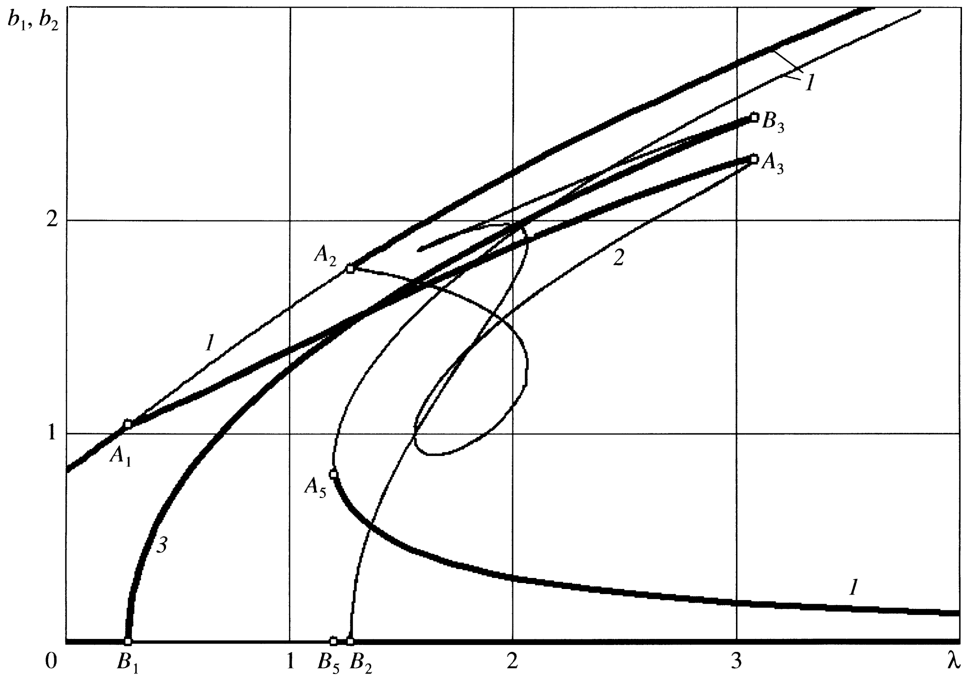

Fig. 2.

Similarly, after substituting solution (2.10), the determinants of the matrices (2.11) take the form

$$\det G_{12} = -\frac{\gamma f_1 f_2 b_2^2}{8\xi(b_2)} - \frac{\gamma f_1^3 b_2}{8\xi^2(b_2)} + \frac{f_1 f_2}{4}$$

$$\det G_{21} = \left(\frac{3\gamma f_1}{4\xi(b_2)} + \frac{\xi^2(b_2)}{2f_1}\right)\left(\frac{3}{4}\gamma b_2 + \frac{1}{2}\frac{f_2}{b_2^2}\right) - \frac{9\gamma^2 f_1 b_2}{16\xi(b_2)}$$

and when $f_1 \to 0$ we obtain the same results as in the previous case.

For small values of the frequency difference there is a unique solution corresponding to the motion of sections in the action plane of the load. When the frequency increases this solution either changes smoothly into three-dimensional motion in the direction of the load, or into three-dimensional motion in a direction opposite to the load. Both these motions are equiprobable when $f_1 = 0$.

If the frequency difference exceeds the value given by formulae (2.12), four oscillation modes are possible in the system. Two of these motions are in the excitation plane of oscillations with "large" and "small" amplitudes and two three-dimensional motions with rotation of the middle line of the rod in two opposite directions. To the left of the point A_2 (the point B_2) three, two and one oscillation modes are possible as the frequency decreases.

The two-dimensional solution, represented by curves *1*, *2* and *3* in Fig. 1, corresponds to the classical solution of the problem of the forced oscillations of a rod in a plane.[1] It can be obtained by solving the second equation of (1.3) with $\nu = 0$ or from the solution of system (2.2) with $f_1 = 0$ and $b_1 = 0$. In this case every solution corresponding to curves *1*, *2* and *3* is stable.

The results of a numerical solution of system of equations (2.2), obtained taking dissipation into account, are shown in Fig. 2. The value of the dissipation parameter $\eta = 0.2$, and the values of the remaining parameters are the same as in the calculations, the results of which are shown in Fig. 1. Curve *1* shows the relation $b_2(\lambda)$ for the two-dimensional form of motion, and the zero value of $b_1(\lambda)$ corresponds to it. This solution is unstable between the points A_1 (B_1) and A_2 (B_2). The point A_4 (B_4), corresponding to the maximum amplitude of the two-dimensional form of motion, is not shown. The relations $b_2(\lambda)$ and $b_1(\lambda)$ of the three-dimensional form correspond to curves *2* and *3*. The presence of friction forces leads to finite values of the maximum amplitudes of the oscillations, and the three-dimensional form of the oscillations is largely quenched.

When the frequency increases, separation occurs on one of the stable branches of the two-dimensional form *1* at the point A_3 (B_3). When the frequency is reduced separation of the oscillations with the two-dimensional form of motion into the three-dimensional form occurs at the points A_2 (B_2) or A_5 (B_5). If dissipation is taken into account the branching point of the solutions A_2 (B_2) is shifted towards a reduction in frequency and stabilization of the solution between points A_2 (B_2) and A_4 (B_4).

As follows from formulae (2.12), a reduction in the parameter δ shifts the stable part of the two-dimensional solution into the high-frequency and high-amplitude region, so that for a rod with equal cross-section moments of inertia, for example, a circular cross-section, this motion is impossible.

When oscillations are excited in the plane of lower flexural stiffness, which corresponds to negative values of the parameter δ, the ordinate b_2 of the point A_2 is negative. The point is situated on curve *6* (Fig. 1), corresponding to the two-dimensional beyond-resonance solution and is not a point of a change of stability, and only a change in the degree of stability of the solution occurs in it. However, the point occurs in the solutions, denoted in Fig. 1 by *7* and *8*, to the right of which the solution is stable. Even when there is no dis-

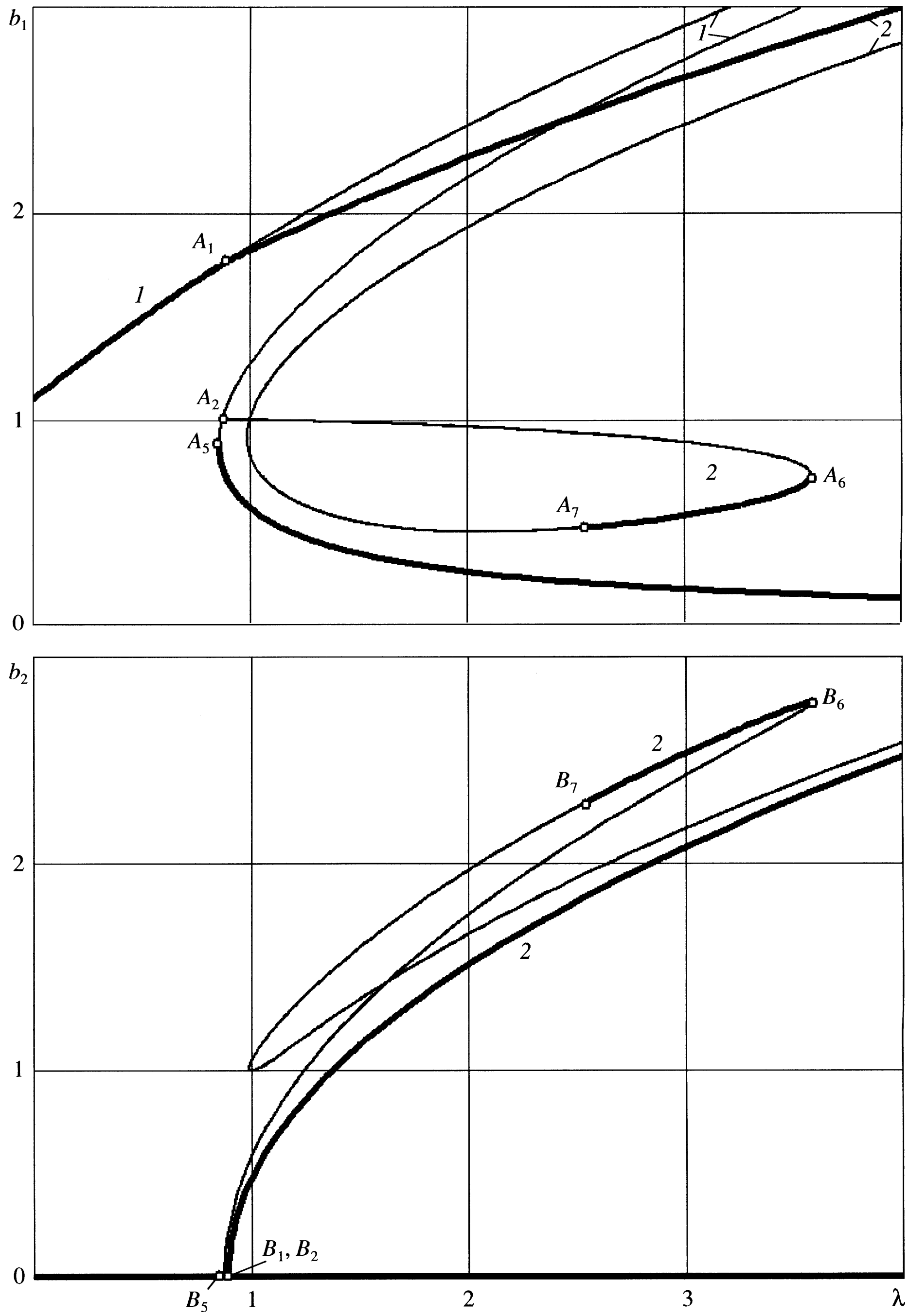

Fig. 3.

sipation, it is not possible to obtain an analytical expression for the coordinates of this point in view of the complexity of the formulae obtained.

The results of a numerical solution of system of equations (2.2) for

$$f_1 = 1, \quad f_2 = 10^{-6}, \quad \eta = 0.05, \quad \psi_1 = 0, \quad \delta = 1$$

are shown in Fig. 3 and correspond to the excitation of oscillations in the plane of lower stiffness and a small value of the dissipation parameter. The maximum amplitudes of curves *1* and *2* are very large and are not shown here. The part of curve *2* between the points A_7 (B_7) and A_6 (B_6) corresponds to the stable solution. In this case, points on the middle line of the rod move along an ellipse mainly in a plane orthogonal to the direction of the load. This stable solution does not exist for large values of the dissipation parameter.

3. Superharmonic oscillations

Consider oscillations of the rod in the neighbourhood of the first superharmonic resonance. In this case we will assume that the amplitudes of the inducing force on the right-hand side of Eqs (1.4) are finite: $f_k(t) = f_k \cos \mu t$, while the frequency $\mu = (1 + \varepsilon\lambda)/3$. We will

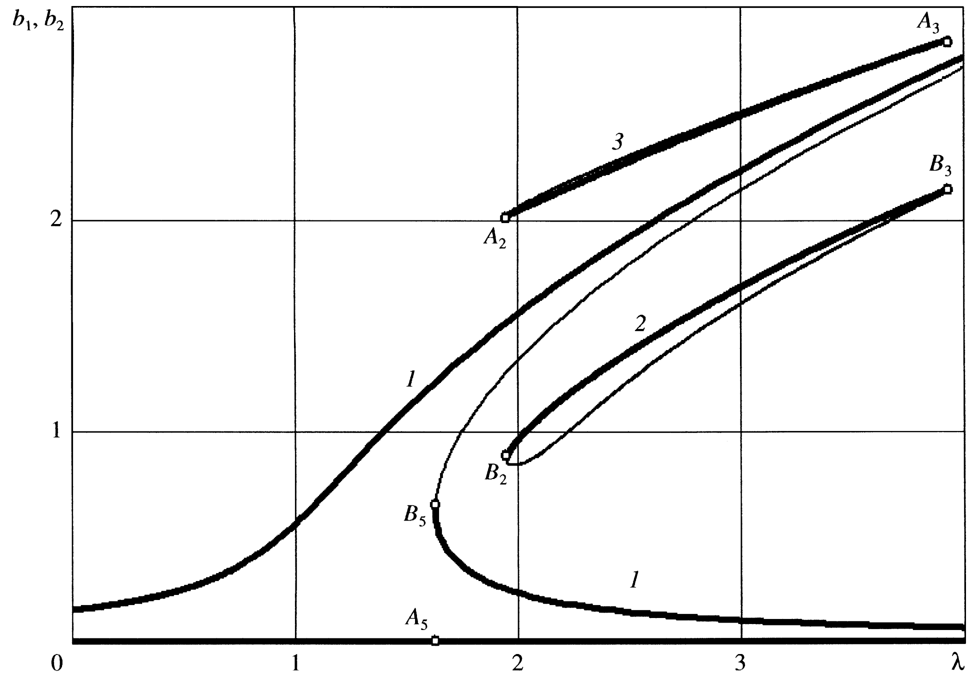

Fig. 4.

seek a solution of system (1.4) in the form of the sum of two harmonics

$$\varphi_k = b_k\cos(t+\alpha_k) + \Delta_k\cos(\mu t+\psi_k), \quad k = 1, 2; \quad \Delta_k = \frac{9}{8}f_k \tag{3.1}$$

The first term corresponds to the third harmonic of the oscillations of the system with a frequency close to unity, while the second term corresponds to the first harmonic of the oscillations, which, far from the fundamental resonance of the system, is determined by the linear theory.

Substituting expressions (3.1) into system (1.4) and using the averaging method, we obtain the following system of equations in slow variables

$$\begin{aligned}\dot{b}_k &= -\frac{1}{8}\varepsilon\gamma b_1^k b_2^{3-k}\sin 2\Delta\alpha_k + \frac{1}{8}\varepsilon\Delta_k^3\sin(\alpha_k - 3\psi_k) + \\ &+ \frac{1}{2}\varepsilon\Delta_1\Delta_2 b_{3-k}\sin\Delta\alpha_k\cos\Delta\psi - \frac{1}{2}\varepsilon\eta b_k \\ \dot{\alpha}_k &= \frac{1}{2}\varepsilon[\delta(k-1) - 2\lambda] + \frac{3}{8}\varepsilon\gamma b_k^2 + \frac{1}{8}\varepsilon\gamma b_{3-k}^2[2+\cos 2(\alpha_1-\alpha_2)] + \\ &+ \frac{1}{4}\varepsilon\gamma(3\Delta_k + \Delta_{3-k}) + \varepsilon\gamma\Delta_1\Delta_2\frac{b_{3-k}}{2b_k}\cos\Delta\alpha\cos\Delta\psi + \frac{1}{8}\varepsilon\gamma\Delta_k[\Delta_k^2\cos(\alpha_k - 3\psi_k) + \\ &+ \varepsilon\Delta_{3-k}^2\cos(\alpha_{3-k} - \psi_k - 2\psi_{3-k})]; \quad \Delta\alpha_k = \alpha_{3-k} - \alpha_k, \quad \Delta\psi = \psi_2 - \psi_1, \quad k = 1, 2\end{aligned} \tag{3.2}$$

We will consider the solution of system (3.2) when there is no dissipation. For values of the phases of the external load and the phase additions

$$\psi_1 = \pi/2, \quad \psi_2 = 0, \quad \alpha_1 = \pi/2, \quad \alpha_2 = 0$$

the first two equations are converted into an identity, while the last two take the form

$$-\lambda + \frac{1}{2}(k-1)\delta + \frac{3}{8}\gamma b_k^2 + \frac{1}{8}\gamma b_{3-k}^2 + \frac{1}{4}\gamma(3\Delta_k^2 + \Delta_{3-k}^2) + \frac{(-1)^k}{4}\gamma\Delta_k\frac{(\Delta_k^2 - \Delta_{3-k}^2)}{2b_k} = 0,$$
$$k = 1, 2 \tag{3.3}$$

and, by the replacement of variables

$$\delta* = \delta + \gamma(\Delta_2^2 - \Delta_1^2), \quad \lambda* = \lambda - \frac{1}{4}\gamma(\Delta_2^2 + 3\Delta_1^2), \quad f_k^* = \frac{(-1)^k}{4}\gamma\Delta_k(\Delta_k^2 - \Delta_{3-k}^2) \tag{3.4}$$

are reduced to the form (2.4). We obtain a second solution of system (3.2) for zero values of the phases of the external load and of the phase additions: $\psi_1 = \psi_2 = \alpha_1 = \alpha_2 = 0$. In the limit as $f_1 \to 0$, this solution is identical with the solution of system (3.3) everywhere, apart from the point $b_2 = f_2^*/\delta^*$. The amplitude-frequency characteristics of the superharmonic resonance when there is no dissipation qualitatively coincide with those shown in Fig. 1, taking replacement (3.4) into account.

Just as for forced oscillations of the rod in the neighbourhood of the principal resonance, there are two-dimensional and three-dimensional solutions at the first superharmonic resonance. Since $\delta^* > \delta$, the stable part of the two-dimensional solution is increased, and the point A_2 (B_2) is situated close to the point A_1 (B_1). Moreover, oscillations in the plane of action of the external load are possible for zero and negative values of the parameter δ, which corresponds to a rod of circular cross-section ($\delta = 0$) and excitation of the oscillations in the plane of least flexural stiffness of the rod ($-\gamma\Delta_2^2 < \delta < 0$).

To investigate the effect of dissipative forces we solved system of equations (3.2) numerically by analytical extension with respect to the parameter. As in the case of two-dimensional oscillations,[9] the phenomenon of superharmonic resonance is only possible for comparatively low values of the dissipation parameter. In the first place the forces of viscous friction suppress the three-dimensional form of motion. In Fig. 4 we show graphs of $b_1(\lambda)$ and $b_2(\lambda)$ for the following values of the system parameters:

$$f_1 = 10^{-6}, \quad f_2 = 1, \quad \eta = 0.06, \quad \psi_1 = 0, \quad \delta = 1/2$$

The values of all the parameters, apart from the dissipation parameter, are the same as in the calculations, the results of which are shown in Figs 1 and 2 for the fundamental resonance. Curve *1* corresponds to the amplitude $b_2(\lambda)$ of the two-dimensional solution, and curves *2* and *3* correspond to the amplitudes $b_2(\lambda)$ and $b_1(\lambda)$ of the three-dimensional solution. An isolated part of the three-dimensional form of the motion of the rod exists for $0.012 < \eta < 0.074$; for large values of the dissipation parameter there is no three-dimensional form of motion. Similar isolated parts also exist for oscillations in the neighbourhood of the principal resonance.[3]

There are several stable modes of motion for forced oscillations of the rod with close values of the principal flexural stiffnesses in the neighbourhood of the principal and first superharmonic resonances. Some of these are isolated, and external perturbations are necessary to obtain them.

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